Higher rank groups have fixed price one

(joint w/ Mikołaj Frączyk and Amanda Wilkens)

(also joint work with Miklós Abért)

Theorem [FMW]

Higher rank groups have fixed price one

(That is, their ess. free pmp actions have cost one)

Examples: $SL_3(\mathbb{R})$, $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, $Aut(T) \times Aut(T)$, $Isom(\mathbb{H}^3) \times Aut(T)$

Defn $\Gamma < G$ is a lattice if it is discrete and $G \curvearrowright G/\Gamma$ has an invariant (Borel) prob. measure.

For G ctble, this just means finite index

Two flavours: if G/Γ is compact, we call Γ cocompact/uniform.

Furman: Lattices in higher rank simple Lie G are ME Rigid

$\Gamma < G$ lattice, and $\Delta \sim_{ME} \Gamma$, then Δ/F is isom to a lattice in G

Burger-Mozes: First examples of finitely pres. simple torsionfree groups (special lattices in $Aut(T) \times Aut(T)$)

$\Gamma < Isom(\mathbb{H}^d)$ torsion free lattice $\longleftrightarrow$ finite vol. complete hyperbolic d-manifold $\Gamma \backslash \mathbb{H}^d$

Corollary (Immediate)

Any lattice in a higher rank group has fixed price one

Known for some lattices, because they have "right angled" generating sets. And they don't always exist

Defn The rank of Γ is $d(\Gamma)$ min size of a generating set

Given $\Gamma < G$ a lattice, we can write $G = \bigsqcup_{\gamma \in \Gamma} \gamma \mathcal{F}$ for $\mathcal{F} \subset G$ Borel

Defn $\mathrm{covol}(\Gamma) = \lambda(\mathcal{F})$, where λ is Haar measure for G

Corollary (FMW + Abért-M or Carderi)

G higher rank simple Lie group

$\Gamma_n < G$ has $\operatorname{covol}(\Gamma_n) \to \infty$, then the rank gradient

$$RG(G, (\Gamma_n)) = \lim_{n\to\infty} \frac{d(\Gamma_n) - 1}{\operatorname{covol}(\Gamma_n)}$$ is zero

$H \leq_{f.i} \mathbb{F}_n$ Then $d(H) - 1 = [\mathbb{F}_n : H] \cdot (n-1)$

So for <u>any</u> f.g group Γ and $H \leq_{f.i} \Gamma$, we have

$$\frac{d(H) - 1}{[\Gamma : H]} \leq d(\Gamma) - 1$$

<u>Defn</u> A chain of subgroups in $\Gamma = \Gamma_0 > \Gamma_1 > \Gamma_2 > \cdots$

Theorem (Lackenby + Abért - Jaikin - Zapirain - Nikolov)

$\Gamma = \langle S \rangle$ finitely pres., (Γ_n) chain of normal f.i. subgroups $\bigcap_n \Gamma_n = 1$

Then at least one of the following is true:

(1) for large n, Γ_n is a nontrivial free product

(2) $\mathrm{Cay}(\Gamma/\Gamma_n, S)$ is an expander seq.

(3) $RG(\Gamma, (\Gamma_n))$ is zero

Moral positive rank gradient is win-win.

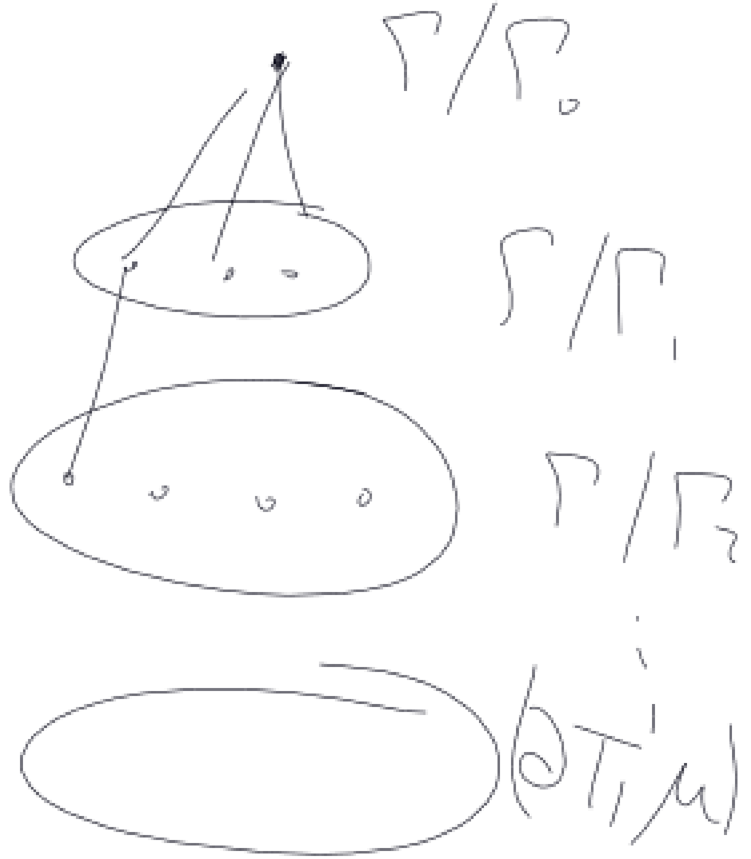
Theorem [Abért-Nikolov]

If $\Gamma = \Gamma_0 > \Gamma_1 > \dots$ is a "Farber" chain of f.i. subgroups.

Then $RG(\Gamma, (\Gamma_n)) = \mathrm{cost}(\Gamma \curvearrowright \partial T) - 1$

T is the coset tree

Farber means $\Gamma \curvearrowright (\partial T, \mu)$ ess. freely.



How to prove fixed price one (for $G \times \mathbb{Z}$ and higher rank G)

- If $G \curvearrowright (X,\mu) \xrightarrow{f} G \curvearrowright (Y,\nu)$ is a factor, then

  $\mathrm{cost}_G(X,\mu) \leq \mathrm{cost}_G(Y,\nu)$

- Also true for "weak factors"; ∃ seq of factors f_n weakly converging
- Free actions weakly factor onto their "Bernoulli extensions"

  (eg. for $\Gamma \curvearrowright (X,\mu)$, ↗ is $\Gamma \curvearrowright X \times [0,1]^\Gamma$)
- weak factoring is transitive
- Bern. ext. weakly factor onto a "special action"
- special action has cost one